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Disjointness Graphs

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Abstract.

The *disjointness graph* of a collection of objects is defined such that two objects are adjacent if they are "disjoint". We propose a framework for studying these graphs when the notion of disjointness is topologically defined via a bounded meet-semilattice, in which case the adjacency matrix of the disjointness graph admits a canonical decomposition in terms of the Möbius function and principal upper sets of the bounded meet-semilattice. Moreover, if this semilattice admits a group action that is multiplicity-free on its upper fiber, then we give a canonical "inclusion-exclusion-type" formula for the spectrum of the disjointness graph. As a corollary, we give good expressions for the eigenvalues of disjointness graphs of a few finite permutation groups in a unified way.

When the semilattice is a Cameron–Deza permutation geometry, the canonical decomposition of its disjointness graph, also known as the *permutation derangement graph*, is related to a normalization (integral form) of *the forgotten symmetric functions* under Frobenius' characteristic map. In his seminal monograph on the subject, Macdonald remarks that this basis has "no particularly simple direct description" and has since been largely forgotten in the symmetric function literature. Our results suggest that forgotten bases can provide simpler expressions for character sums over normal sets of group elements that admit simple combinatorial/topological descriptions.

Keywords: Möbius Inversion, Representation Theory, Derangements, Algebraic Graph Theory, Association Schemes, Extremal Combinatorics, Group Theory, Posets.

To the memory of Ian G. Macdonald

1 Introduction

The *disjointness graph* of a collection of objects is defined such that two objects are adjacent if they are "disjoint". The prototypical example is the *Kneser graph*, that is, the disjointness graph of the set of all k-element subsets of a n-element set, a cornerstone of algebraic and topological combinatorics. Disjointness graphs play a key role in extremal combinatorics, in particular, the field of *Erdős–Ko–Rado (EKR) combinatorics*, as their independent sets are "intersecting" families, by design. Many seminal results in

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extremal combinatorics amount to characterizing the maximum independent sets of an infinite family of disjointness graphs, defined over a variety of different combinatorial domains and for varying notions of "disjointness". Spectral methods have played a vital role in the development of this area, which has been masterfully expounded in Godsil and Meagher's textbook on the subject [4].

However, what it means for two objects to be "disjoint" is of course vague and subject to interpretation. To make this a proper definition, we begin with the universe of finite *meet-semilattices* $(X, \leq)$ with a unique bottom element $\emptyset$ so that two top elements $x, y \in X$ *a priori* are *disjoint* if $x \wedge y = \emptyset$. In this way, we associate to each such meet-semilattice a unique disjointness graph defined on its top elements (see Section 2). Most, if not all of the EKR-type results in the literature fit into this framework, and a goal of this work is to give good general expressions for the eigenvalues of disjointness graphs for meet-semilattices that are sufficiently regular (e.g., Theorem 2 and Theorem 3). We would be remiss not to mention previous order-theoretical frameworks for computing eigenvalues of (disjointness) graphs in P-polynomial association schemes, e.g., Delsarte's *regular semilattices* [3], Stanton's *harmonic posets* [14], and Terwilliger's *uniform posets* [16]. In this work we assume less regularity on the poset, which allows us to consider the disjointness graphs of meet-semilattices associated with other finite spaces such as permutation groups that generally do not belong to any of the aforementioned frameworks.

In the permutation group setting, the eigenvalues of disjointness graphs, also known as *derangement graphs*, not only have applications to extremal combinatorics, but also to classic factorization problems in groups. Larsen, Shalev, and Tiep [7] have recently shown that any element in a sufficiently large transitive finite simple permutation group is a product of two derangements. They conjecture this should hold for all finite simple groups, but their character estimates are too crude to deduce this. In Section 4, we show our techniques give a simple proof that for all n, any element of the symmetric group S_n is a product of two derangements (in fact, $\Omega(n!)$ pairs of derangements). Whether our methods can be extended to finite simple permutation groups is left for future work.

This work can be seen as a conceptual companion paper to [8], which considers in much detail the eigenvalues of the permutation and perfect matching derangement graphs.

2 Posets, Incidence Algebras, and Disjointness Graphs

In this section, we formally define the disjointness graph of a bounded meet-semilattice and give a canonical decomposition of its adjacency matrix that we call its *zeta decomposition*. We first recall some order-theoretic preliminaries (see [6] for more details).

A *partially ordered set* or *poset* $(X, \leq)$ is a set X and a binary relation $\leq$ that satisfies $x \leq x$ for all $x \in X$; if $x \leq y \leq x$, then $x = y$; and if $x \leq y \leq z$, then $x \leq y$. We write $x < y$ if $x \leq y$ and $x \neq y$. We say that y *covers* x if $x < y$ and there exists no z such that $x < z < y$. A set $Y \subseteq X$ is a *principal upper set* if $Y = \{y \in X : x \leq y\}$ for some $x \in X$. We say $x \in X$ is a *top element* if no other element covers it, and we say $x \in X$ is a *bottom element* if it covers no other element. Let $\bar{X} \subseteq X$ be the set of top elements. Two elements are *incomparable* if both $x \nleq y$ and $y \nleq x$. Two elements $x, y \in X$ have a *meet* if there exists an element $x \wedge y \in X$ such that $x \wedge y \leq x$, $x \wedge y \leq y$, and if both $w \leq x$ and $w \leq y$, then $w \leq x \wedge y$ for all $w \in X$. A *meet-semilattice* is a poset such that all meets exist. A meet-semilattice is *bounded* if there exists a unique $\emptyset \in X$ such that $x \wedge \emptyset = \emptyset$ for all $x \in X$. Any finite poset $(X, \leq)$ can be made into a bounded meet-semilattice $(X \cup \{\emptyset\}, \leq)$ by setting $\emptyset \leq x$ for all $x \in X$.

A *graded poset* $(X, \leq)$ is a poset where all the maximal chains have the same length. Graded posets are equipped with a *rank function* r from X to the natural numbers inductively defined such that $r(\bar{x}) = 0$ for all top elements $\bar{x} \in \bar{X}$, and if y covers x then $r(y) = r(x) + 1$. The *rank* of a graded poset $r(X, \leq)$ is the maximum rank over its elements. Let $X_k := \{x \in X : r(x) = k\}$ be the *fiber* of rank k elements of X. The rank of the elements of the top fiber equals the rank of the poset, i.e., $X_{r(X,\leq)} = \bar{X}$.

The elements of the *incidence algebra* $\mathcal{I}(X, \leq)$ of a poset $(X, \leq)$ are functions f assigning to each nonempty interval $[x, y]$ a scalar $f(x, y)$ drawn from a commutative ring with unity. Addition $+$ is defined pointwise and multiplication $*$ is *convolution*, i.e.,

$$(f * g)(x, z) := \sum_{x \leq y \leq z} f(x, y) g(y, z)$$

for all $f, g \in \mathcal{I}(X, \leq)$ and $x, z \in X$. Any incidence algebra can be realized as an upper-triangular matrix algebra as follows. To each function $f \in \mathcal{I}(X, \leq)$ we associate a $|X| \times |X|$ matrix A such that $A_{x,y} := f(x, y)$ whenever $x \leq y$, and is 0 otherwise. Reordering the indices of A with respect to any total ordering that extends the partial order $\leq$ puts A into upper-triangular form such that its zero-pattern above the diagonal is determined by the incomparable elements of $(X, \leq)$.

The *identity function* ι of an incidence algebra is defined such that $\iota(x, y) := 1$ if $x = y$, and is 0 otherwise. The *zeta function* of an incidence algebra is the support of the comparable elements, i.e., the constant function $\zeta(x, y) = 1$ for every nonempty interval $[x, y]$. The *Möbius function* of a poset is recursively defined for all pairs $x \leq z \in X$ as

$$\mu(x, z) := -\sum_{x \leq y < z} \mu(x, y)$$

where $\mu(x, x) := 1$. It is well-known that the Möbius function is the inverse of the zeta function, that is,

$$\mu\zeta = \zeta\mu = \iota.$$

Since the zeta function is represented by an upper unitriangular matrix with integer entries, the Möbius function is also represented by an upper unitriangular matrix with integer entries. In particular, we have

$$\mu\zeta(x,z) = [\mu * \zeta](x,z) = \sum_{x \le y \le z} \mu(x,y)\zeta(y,z) = \sum_{x \le y \le z} \mu(x,y) = 0 \tag{2.1}$$

for all $x < z$. For any bounded meet-semilattice $(X, \le)$, let $\mu_x := \mu(\emptyset, x)$ for all $x \in X$.

For any $x \in X$, define the *incidence vector* $w_x \in \mathbb{R}^{|\bar{X}|}$ such that $(w_x)_z = \zeta(x,z)$. Let W_k be the $|X_k| \times |\bar{X}|$ *incidence matrix* defined such that $(W_k)_{y,z} = \zeta(y,z)$ for all $y \in X_k$ and $z \in \bar{X}$. For each $x \in X$, define its *zeta matrix* as $Z_x := w_x w_x^\top$. For any bounded graded meet-semilattice $(X, \le)$, let $Z_k := W_k W_k^\top$ be the *kth zeta matrix* (c.f. *Reimann matrix* [3]). Note that $(Z_x)_{y,z} = 1$ if and only if $x \le y$ and $x \le z$, thus

$$(Z_k)_{y,z} = \left(\sum_{x \in X_k} Z_x \right)_{y,z} = \sum_{x \in X_k} \zeta(x,y)\zeta(x,z) = |\{x \in X_k : x \le y \text{ and } x \le z\}|.$$

Observe that $Z_0 = Z_\emptyset = J$ where J is the $|\bar{X}| \times |\bar{X}|$ all-ones matrix. We are now in a position to define the disjointness graph of $(X, \le)$ and its canonical zeta decomposition.

Definition 1 (Disjointness Graph). *For any bounded meet-semilattice $(X, \le)$, its* disjointness graph *$\Phi := (\bar{X}, E)$ is defined such that $xy \in E$ if $x \wedge y = \emptyset$.*

Theorem 1 (Zeta Decomposition). *Let Φ be the disjointness graph of a bounded meet-semilattice $(X, \le)$. Then we have*

$$\Phi = \sum_{x \in X} \mu_x Z_x.$$

Proof. If $y, z \in \bar{X}$ are disjoint, then $\zeta(w,y)\zeta(w,z) = 0$ for all $w \neq \emptyset$, and so $\Phi_{y,z} = 1$. If $y, z \in \bar{X}$ are not disjoint, then $y \wedge z \neq \emptyset$. Since $\zeta(w,y)\zeta(w,z) = 1$ if and only if $w \in [\emptyset, y \wedge z]$, Equation (2.1) gives

$$\Phi_{y,z} = \sum_{x \in X} \mu_x (Z_x)_{y,z} = \sum_{\emptyset \le x \le y \wedge z} \mu_x (Z_x)_{y,z} = \sum_{\emptyset \le x \le y \wedge z} \mu_x = 0,$$

as desired. □

The zeta decomposition already gives an upper bound on the rank of Φ, but it is indeed difficult to deduce further information about the eigenvalues of Φ without assuming some form of combinatorial regularity on $(X, \le)$. For instance, we say that the Möbius function μ of a bounded graded meet-semilattice $(X, \le)$ is *rank-invariant* if $\mu_x = \mu_y$ for all $x, y \in X_k$ and $0 \le k \le r(X, \le)$. Bounded graded meet-semilattices with rank-invariant Möbius functions admit simpler zeta decompositions:

$$\Phi = \sum_{k=0}^{r(X,\le)} \mu_k \, Z_k. \tag{2.2}$$

Moreover, the eigenvalues of zeta matrices are much easier to compute when $(X, \leq)$ has combinatorial regularity (and there are fewer zeta matrices); however, the eigenspaces of the zeta matrices are typically misaligned. In this case, the relation between the spectrum of the individual Z_k's and their sum is non-linear, and a careful analysis is needed to say anything definitive about the eigenvalues of Φ. On the other hand, if we assume that the eigenspaces of the zeta matrices are *aligned*, i.e., they have a common system of orthogonal eigenvectors $\{v_\eta\}_\eta$, then we get a canonical eigenvalue expression:

$$\eta(\Phi) = \sum_{k=0}^{r(X,\leq)} \mu_k \, \eta(Z_k) \tag{2.3}$$

where $\eta(\cdot)$ denotes the matrix's eigenvalue corresponding to v_η. Despite appearances, this scenario is sufficiently general to include many interesting groups and Gelfand pairs not included in previous frameworks. Since the eigenvalues of the zeta matrices are non-negative, this gives an "inclusion-exclusion-type" formula for the eigenvalues of Φ.

In the next section we continue with this regularity assumption and restrict our attention to the special case of bounded graded meet-semilattices whose zeta matrices belong to *Bose–Mesner algebras* of *conjugacy-class association schemes* of permutation groups.

3 Eigenvalues of Derangement Graphs

For permutation groups G, we say that two elements $g, h \in G$ are "disjoint" if they are *derangements* of one another, i.e., gh^{-1} has no fixed points. In this section, we give a general expression for the eigenvalues of Φ_n with respect to any infinite family of permutation groups $\{G_n\}_{n=0}^{\infty}$ with the property that $G_n \leq G_{n+1}$.

For each n, we define a bounded graded meet-semilattice $(X(n), \leq)$ such that $X_k(n)$ is the set of G_n-double-translates of the cosets G_n/G_{n-k} for all $0 \leq k \leq n$, and $y \in X_{k+1}(n)$ covers $x \in X_k(n)$ if $y \subseteq x$. We abuse notation by setting $X := X(n)$ when n is clear from context. Since G_n acts transitively on each fiber, its Möbius function μ is rank-invariant. We say that $g \in G_n$ is an *abstract derangement* if it does not lie in any conjugate of G_{n-1}, equivalently, $\mathrm{id} \wedge g = \emptyset$. Note that the disjointness graph Φ_n of $(X, \leq)$ is the Cayley graph of G_n generated by its abstract derangements. By Theorem 1, its zeta decomposition is $\Phi_n = \sum_{k=0}^{n} \mu_k \, Z_k$.

Since D_n is normal, i.e., closed under conjugation by G_n, the graph Φ_n belongs to the Bose–Mesner algebra of G_n's conjugacy-class association scheme (see [5], for example). Its eigenvalues $\eta_i(\Phi)$ are indexed by the set $\mathrm{Irr}(G_n)$ of inequivalent ordinary irreps ρ_i of G_n. Let $\rho(k) := 1\!\uparrow_{G_{n-k}}^{G_n}$ be the permutation representation of G_n acting on cosets G_n/G_{n-k}. Let $m_i(k)$ be the multiplicity of ρ_i in $\rho(k)$, and let χ_i be its character. Its *degree* $d_i := \chi_i(1)$ divides $|G_n|$, thus the *co-degree* $\bar{d}_i := |G_n|/\chi_i(1)$ is a positive integer.

Theorem 2 (Eigenvalues of the Abstract Derangement Graph)**.** *For all $\rho_i \in \mathrm{Irr}(G_n)$, we have*

$$\eta_i(\Phi_n) = \bar{d}_i \sum_{k=0}^{n} \mu_k \, \frac{m_i(k)}{[N(G_{n-k}) : G_{n-k}]}$$

where $N(G_{n-k})$ is the normalizer of G_{n-k} in G_n.

Proof. By Frobenius Reciprocity we have

$$m_i(k) = \langle 1 \uparrow_{G_{n-k}}^{G_n}, \chi_i \rangle_{G_n} = \langle 1_{G_{n-k}}, \chi_i \downarrow_{G_{n-k}}^{G_n} \rangle_{G_{n-k}} = \frac{1}{|G_{n-k}|} \sum_{g \in G_{n-k}} \chi_i(g).$$

Without loss of generality, we may assume that the first column of the orthogonal projector E_i onto the ρ_i-isotypic component is the normalized irreducible character $\frac{d_i}{|G_n|}\chi_i$. Without loss of generality, we may also assume that the first row of W_k is indexed by $\mathrm{id} \in G_n$ so that the support of the first row is indexed by all conjugates of G_{n-k}. There are $[G_n : N(G_{n-k})]$ such conjugates. The first row of $W_k^\top$ is the characteristic vector of G_{n-k}, which gives us

$$\left(W_{k,n}^\top E_\nu \right)_{1,1} = \frac{|G_{n-k}|}{|G_n|} \, d_i m_i(k).$$

This identity holds for all subgroups isomorphic to G_{n-k}, hence

$$(Z_k E_i)_{1,1} = \frac{|G_{n-k}|}{|N(G_{n-k})|} \, d_i m_i(k).$$

Dividing by $(E_i)_{1,1} = d_i^2/|G_n|$ gives us

$$Z_k E_i = \frac{|G_n| m_i(k)}{d_i [N(G_{n-k}) : G_{n-k}]} E_i = \frac{\bar{d}_i m_i(k)}{[N(G_{n-k}) : G_{n-k}]} E_i \, ,$$

which finishes the proof. $\square$

The degrees d_i and multiplicities $m_i(k)$ can be computed combinatorially provided there exists a so-called *branching rule* for G_n that describes the restriction of the G_n-irreducible $\rho_i(k)$ into G_{n-1}-irreducibles. This relation is encoded by a graded poset called the *Bratteli diagram* of $\{G_n\}_{n=0}^{\infty}$ whose elements of rank k are the G_k-irreducibles. These posets often posess a great deal of combinatorial regularity, making it possible to enumerate all the paths from the trivial representation $\rho_1(k)$ of G_k to a given irreducible $\rho_i(n)$. We demonstrate this by determining the eigenvalues of the derangement graphs for the Coxeter group of type B_n. Closed-form expressions for the eigenvalues of the derangement graph of type A_{n-1}, i.e., *the permutation derangement graph*, were determined in [8].

3.1 Derangement Graphs of Finite Coxeter Groups

Let $G_n = A_{n-1} \cong S_n$, the *symmetric group* on n symbols, in which case the semilattice $(X, \leq)$ is a *Cameron–Deza permutation geometry* [1]. The Bratteli diagram in this case is *Young's lattice* $\mathbb{Y}$, the quintessential example of a *differential poset* [12], which gives us

$$\eta_\lambda(\Phi_n) = \frac{n!}{f^\lambda} \sum_{k=0}^{n} \frac{(-1)^k}{k!} m_\lambda(k) = \frac{n!}{f^\lambda} \sum_{k=0}^{n} \frac{(-1)^k}{k!} f^{\lambda/(n-k)}$$

where f^λ counts the number of *standard Young tableaux* of shape λ (i.e., the number of paths of the form $\varnothing \rightsquigarrow \lambda$ in Young's lattice) and $f^{\lambda/(n-k)}$ counts the number of *skew standard Young tableaux* of skew shape $\lambda/(n-k)$, i.e., the number of paths of the form $(n-k) \rightsquigarrow \lambda$ in Young's lattice. By the Lindström–Gessel–Viennot lemma [13], these enumerations can be carried out efficiently via determinants, which bodes well for a 'good' combinatorial interpretation. The determinantal identity for f^λ is particularly simple, admitting a nice closed form known as *the hook length formula* $f^\lambda = n!/H_\lambda$ where $H_\lambda = \bar{d}_\lambda$ is the product of the hook lengths of the cells of λ. Thus we have

$$\eta_\lambda(\Phi_n) = H_\lambda \sum_{k=0}^{n} \frac{(-1)^k}{k!} f^{\lambda/(n-k)}.$$

A similar result holds for *the hyperoctahedral group* $B_n = \mathbb{Z}_2 \wr S_n$. Following [17, §2.1.2], recall that the B_n-irreducibles are indexed by *double partitions* of n, i.e., ordered pairs of partitions $(\lambda, \bar{\lambda})$ such that $|\lambda| + |\bar{\lambda}| = n$. Let $S^{(\lambda,\varnothing)}$ to be the B_n-irreducible pulled back from S_n-irreducible S^λ under the surjection $B_n \twoheadrightarrow S_n$. Let $\mathbb{C}_\varepsilon$ be the linear representation associated to the character $\varepsilon : B_n \twoheadrightarrow \{\pm 1\}$ where the canonical generators of $\mathbb{Z}_2^n$ act by (-1), and elements of S_n act trivially. Define $S^{(\varnothing,\bar{\lambda})} := S^{(\bar{\lambda},\varnothing)} \otimes \mathbb{C}_\varepsilon$. Moreover, for a double partition $(\lambda, \bar{\lambda})$ of n, define the B_n-irreducible

$$S^{(\lambda,\bar{\lambda})} := \left(S^{(\lambda,\varnothing)} \boxtimes S^{(\varnothing,\bar{\lambda})} \right) \uparrow_{B_\lambda \times B_{\bar{\lambda}}}^{B_n}$$

where $\boxtimes$ denotes the outer tensor product of representations. The branching rule for B_n is simply

$$1 \uparrow_{B_{n-1}}^{B_n} = \bigoplus_{\lambda^+} S^{(\lambda^+,\bar{\lambda})} \oplus \bigoplus_{\bar{\lambda}^+} S^{(\lambda,\bar{\lambda}^+)}$$

where the summations range over all ways of adding an outer corner to λ and $\bar{\lambda}$ respectively. Iterating the branching law gives

$$1 \uparrow_{B_{n-k}}^{B_n} = \bigoplus_{((n-k),\varnothing) \nearrow^k (\lambda,\bar{\lambda})} S^{(\lambda,\bar{\lambda})}$$

where the summation ranges over all ways of successively adding k outer corners to the trivial representation $((n-k), \varnothing)$ of B_{n-k}. Here, the multiplicity $m_{(\lambda,\bar{\lambda})}$ of $(\lambda, \bar{\lambda})$

is simply the number of paths of the form $((n-k),\varnothing) \rightsquigarrow (\lambda,\bar{\lambda})$ in the 2-differential poset $\mathbb{Y}^2 = \mathbb{Y} \times \mathbb{Y}$. It follows that a *standard tableau* of double partition shape $(\lambda,\bar{\lambda})$ is a path $(\varnothing,\varnothing) \rightsquigarrow (\lambda,\bar{\lambda})$ in $\mathbb{Y}^2$, i.e., a filling of the diagrams of λ and $\bar{\lambda}$ with the numbers $1,2,\ldots,|\lambda|+|\bar{\lambda}|$ such that the numbers are strictly increasing along rows and down columns. The number of such standard tableaux $f^{(\lambda,\bar{\lambda})} = \binom{n}{|\lambda|} f^{\lambda} f^{\bar{\lambda}}$, thus

$$m_{\lambda,\bar{\lambda}} = \binom{n}{|\bar{\lambda}|} f^{\lambda/(n-k)} f^{\bar{\lambda}}.$$

The foregoing shows for all double partitions $(\lambda,\bar{\lambda})$ of n that

$$\eta_{\lambda,\bar{\lambda}}(\Phi_n) = \frac{2^n n!}{\binom{n}{|\lambda|} f^{\lambda} f^{\bar{\lambda}}} \sum_{k=0}^{n} \frac{(-1)^k}{2^k k!} \binom{n}{|\bar{\lambda}|} f^{\lambda/(n-k)} f^{\bar{\lambda}/\varnothing} = (n)_{|\bar{\lambda}|} H_{\lambda} \sum_{k=0}^{n} \frac{(-1)^k}{k!} 2^{n-k} f^{\lambda/(n-k)}.$$

The *alternating group* $\mathrm{Alt}_n \trianglelefteq S_n$ and the *demihyperoctahedral group* $D_n \trianglelefteq B_n$ are the unique index-2 normal subgroups of S_n and B_n, respectively. Similar eigenvalue calculations for the derangement graphs of Alt_n and D_n can be carried out *mutandis mutandis* or via Clifford theory, which we defer to a forthcoming full version of this work.

3.2 Derangement Graphs of Finite General Linear Groups

For any prime power q, let $G_n = GL(n,q)$ be the *finite general linear group* over $\mathbb{F}_q$, noting that $G_{n-1} = GL(n-1,q) \cong GL(n-1,q) \times 1 \leq G_n$. The representation theory of the finite general linear group is far more baroque than the "$q \to 1$" case of the symmetric group, so we are only able to provide a rough sketch of our results on $GL(n,q)$, deferring formal proofs to the full version. Our treatment follows Macdonald [9, §IV].

Let $\psi_n(q) := \prod_{i=1}^{n}(q^i - 1)$. Basic counting reveals that $|G_n| = q^{\binom{n}{2}} \psi_n(q)$. Let Θ denote the set of irreducible polynomials of $\mathbb{F}_q[x]$ aside from x, and let Par denote the set of all integer partitions. The irreducibles and conjugacy classes of G_n are indexed by partition-valued functions $\underline{\lambda} : \Theta \to \mathrm{Par}$ such that $\sum_{\varphi \in \Theta} \deg \varphi \cdot |\underline{\lambda}(\varphi)| = n$. For all $\varphi \in \Theta$, let $q_{\varphi} := q^{\deg \varphi}$, and let $n(\lambda) := \sum_{i \geq 1}(i-1)\lambda_i$. The character degrees of G_n admit the following expression, which can be seen as a q-analogue of the hook formula [9, §IV.6]:

$$d_{\underline{\lambda}} = \psi_n(q) \prod_{\varphi \in \Theta} q_{\varphi}^{n(\underline{\lambda}(\varphi)^{\top})} \tilde{H}_{\lambda}(q_{\varphi})^{-1} \quad \text{where} \quad \tilde{H}_{\lambda}(q_{\varphi}) := \prod_{(i,j) \in \lambda} (q_{\varphi}^{h_{i,j}^{\lambda}} - 1).$$

The Möbius number of $(X,\leq)$ is $\mu_k = (-1)^k q^{\binom{k}{2}}$ (here, the "$q \to 1$" case recovers the Cameron-Deza permutation geometry). Theorem 2 gives us

$$\eta_{\underline{\lambda}}(\Phi_n) = \frac{|G_n|}{d_{\underline{\lambda}}} \sum_{k=0}^{n} \frac{(-1)^k q^{\binom{k}{2}}}{|G_k|} m_{\underline{\lambda}}(k).$$

The multiplicity $m_{\underline{\lambda}}(k) =: K_{\underline{\lambda},(n-k,1^k)}$ is given by a combinatorial Pieri-type rule in the representation ring of $GL(n,q)$ which has a Hopf algebra structure. This gives us

$$= q^{\binom{n}{2}} \prod_{\varphi \in \Theta} \frac{\tilde{H}_{\underline{\lambda}(\varphi)}(q_\varphi)}{q_\varphi^{n(\underline{\lambda}(\varphi)^\top)}} \sum_{k=0}^{n} \frac{(-1)^k}{\psi_k(q)} K_{\underline{\lambda},(n-k,1^k)}.$$

When $\underline{\lambda}$ is the trivial irrep, we recover Chen and Rota's formula for the *q-derangements* of $V = \mathbb{F}_q^n$, i.e., the number of $g \in G_n$ such that $gv \neq v\ \forall v \in V \setminus \{0\}$ [2, Ex. 3.2]. Whether the eigenvalues $\eta_{\underline{\lambda}}(\Phi_n)$ of the q-derangement graph admit a nice combinatorial interpretation and closed form analogous to the "$q \to 1$" case [8] is left for future work.

3.3 The t-Disjointness Graph

A natural generalization of the disjointness graph that has been central to extremal combinatorics is the *t-disjointness graph* $\Phi_{n,t}$ defined such that $x, y \in X_n$ are adjacent if $x \wedge y \in X_{<t}$ where $X_{<t} \subseteq X$ is the set of elements of rank less than t. Many seminal results in extremal combinatorics amount to characterizing the maximum independent sets of infinite families of t-disjointness graphs for various combinatorial domains (see [4] for an overview of these results). Note that $\Phi_{n,t}$ is simply the disjointness graph of the quotient $(X/X_{<t}, \leq)$ obtained by identifying all elements of rank less than t with $\emptyset$. Here, the zeta matrices $Z_k^{\geq t}$ of $(X/X_{<t}, \leq)$ are defined such that $Z_0^{\geq t} := J$ and $Z_k^{\geq t} := Z_{k+t-1}$ for all $1 \leq k \leq r(X) - t + 1$. Indeed, this poset retains most of the essential properties of $(X, \leq)$ except that it has a more general Möbius function $\mu_k^{\geq t}$ which can be computed by solving a unitriangular system of linear equations.

For example, in the case where $G_n = S_n$, the Möbius function is $\mu_k = (-1)^k$, and one can show that $\mu_{k-t+1}^{\geq t} = (-1)^{k-t-1}\binom{k-1}{t-1}$ for all $k \geq t$ (see [11, Lemma 31], for example). A *t-derangement* of S_n is an element with less than t fixed points. The following formula for the eigenvalues of the t-derangement graphs of S_n is immediate.

Theorem 3 (Eigenvalues of the t-Derangement Graph of S_n)**.** *For all $\lambda \vdash n$, we have*

$$\eta_\lambda(\Phi_{n,t}) = H_\lambda \left(\frac{f^{\lambda/(n-t+1)}}{(t-1)!} + (-1)^{t-1} \sum_{k=t}^{n} \frac{(-1)^k}{k!} \binom{k-1}{t-1} f^{\lambda/(n-k)} \right).$$

The $t = 1$ case was considered in the companion paper [8], where it was shown that the eigenvalues admit a good combinatorial interpretation with a closed-form expression. For $t \geq 2$, the eigenvalues do not seem to share many of the nice properties that hold in the $t = 1$ case (e.g., the *alternating sign property* [8, Corollary 2]), but we leave a more careful analysis for future work.

4 Products of Derangements

In this section, we give a classical application to factorization in the symmetric group, which we present in the matrix-theoretic language of association schemes [5]. We continue with the same notation and assumptions introduced in the previous section.

Let $A_0, A_1, \cdots, A_d$ be the standard basis for the conjugacy-class association scheme of a group G_n. For a matrix $A = \sum_{i=0}^{d} \alpha_i A_i$, we define $[A_i]A := \alpha_i$ to be the coefficient α_i corresponding to A_i. Let c_i be a representative of the conjugacy class C_i of G_n. It is well-known that the eigenvalue of $\mathrm{Cay}(G_n, C_i)$ corresponding to the ρ-isotypic component equals $|C_i|\chi^\rho(c_i)/\dim\rho$. Let η_ρ be the eigenvalue of $\Phi_n := \mathrm{Cay}(G_n, D_n)$ corresponding to ρ, and let E_ρ denote the orthogonal projection onto the ρ-isotypic component. Then

$$[A_i]\ \Phi_n^2 = \frac{\mathrm{Tr}\left(A_i\ \Phi_n^2\right)}{|G_n||C_i|} = \frac{\mathrm{Tr}\left(\sum_{\rho\in\mathrm{Irr}(G_n)} |C_i|\chi^\rho(c_i)\cdot\eta_\rho^2 E_\rho\right)}{|G_n||C_i|\chi^\rho(1)} = \frac{1}{|G_n|}\sum_{\rho\in\mathrm{Irr}(G_n)} \dim\rho\ \chi^\rho(c_i)\eta_\rho^2.$$

Since $[A_i]\ \Phi_n^2 = 0$ if and only if no $c_i \in C_i$ can be written as a product of two derangements, one aims to show that $\sum_{\rho\in\mathrm{Irr}(G_n)} \dim\rho\ \chi^\rho(c_i)\eta_\rho^2 > 0$ for each conjugacy class C_i. We demonstrate how our techniques make short work of this task when $G_n = S_n$.

Theorem 4. *Any permutation of S_n can be written as a product of $\Omega(n!)$ pairs of derangements.*

In other words, there exists a constant $c > 0$ such that $(\Phi_n^2)_{\pi,\sigma} \geq cn!$ for all $\pi, \sigma \in S_n$.

Proof. Let $h_{i,j}^\lambda$ be the hook length of the cell $(i,j) \in \lambda$. Define $H_\lambda := \prod_{(i,j)\in\lambda} h_{i,j}^\lambda$. Recall that $\chi^\lambda(1) = d_\lambda = n!/H_\lambda$ by the hook length formula, which gives us

$$[A_\mu]\ \Phi_n^2 = \frac{1}{n!}\sum_{\lambda\vdash n} d_\lambda\ \chi^\lambda(\mu)\eta_\lambda^2 = \sum_{\lambda\vdash n}\frac{\chi^\lambda(\mu)}{H_\lambda}\ D_\lambda^2$$

where the D_λ's are the so-called *λ-colored derangements* introduced in [8], where it was also shown that $D_\lambda \leq H^1(\lambda) := h_{1,1}^\lambda h_{1,2}^\lambda \cdots h_{1,\lambda_1}^\lambda$ [8, Proposition 11]. It is well-known that $|\chi^\lambda(\mu)| \leq d_\lambda$ for all $\mu \vdash n$. For any $\lambda = (\lambda_1, \ldots, \lambda_\ell) \vdash n$, let $\lambda' := (\lambda_2, \ldots, \lambda_\ell) \vdash (n - \lambda_1)$. These upper bounds along with the fact that $D_{(n)} = |D_n| < (n!+1)/e$ give us

$$\begin{aligned}
\sum_{\lambda\vdash n}\frac{\chi^\lambda(\mu)}{H_\lambda}\ D_\lambda^2 \geq \frac{n!}{e^2} - \sum_{\substack{\lambda\vdash n\\ \lambda\neq(n)}}\frac{|\chi^\lambda(\mu)|}{H(\lambda')}D_\lambda \geq \frac{n!}{e^2} - \sum_{\substack{\lambda\vdash n\\ \lambda\neq(n)}}\frac{d_\lambda}{H(\lambda')}D_\lambda &\geq n!\left(\frac{1}{e^2} - \sum_{\substack{\lambda\vdash n\\ \lambda\neq(n)}}\frac{1}{H(\lambda')^2}\right)\\
&\geq n!\left(\frac{1}{e^2} - \sum_{k=1}^{n-1}\frac{1}{k!^2}\sum_{\lambda\vdash k} d_\lambda^2\right)\\
&= n!\left[\frac{1}{e^2} - \sum_{k=1}^{n-1}\frac{1}{k!}\right].
\end{aligned}$$

The bracketed sum is slightly negative, so we need a more refined analysis. Noting that $\sum_{k=4} \frac{1}{k!} \approx 0.051615162$ and $e^{-2} \approx 0.135335283$, we claim for all $n \geq 8$ and $\mu \vdash n$ that

$$e^{-2} \left[\sum_{\substack{\lambda \vdash n \\ n-\lambda_1 \leq 3}} \chi^\lambda(\mu) \left(\frac{H^1(\lambda)^2}{n! H_\lambda} \right) \right] - \sum_{k=4}^{n-1} \frac{1}{k!} > 0.$$

It suffices to show that the bracketed summation is no less than $2/5$ for all $n \geq 8$. Since $\chi_{(n-1,1)}(\mu) \geq -1$ for all $\mu \vdash n$, we have

$$\sum_{\substack{\lambda \vdash n \\ n-\lambda_1 \leq 3}} \chi^\lambda(\mu) \left(\frac{H^1(\lambda)^2}{n! H_\lambda} \right) = \sum_{\substack{\lambda \vdash n \\ n-\lambda_1 \leq 3}} \frac{\chi^\lambda(\mu)}{\chi^\lambda(1)} \left(\frac{1}{H(\lambda')^2} \right) \geq 1 - \frac{1}{n} - \frac{2}{4} - \frac{1}{9} - \frac{2}{36} > \frac{2}{5},$$

which completes the proof. □

For $n \leq 7$, one can verify in `SAGE` [15] that the entries of Φ_n^2 are nonzero. Similar results hold for other permutation groups, which we consider in future work.

5 Afterword: The Forgotten Combinatorial Basis

Our results on S_n translate to results on symmetric functions via Frobenius' characteristic map. In particular, the zeta matrices Z_k in the zeta decomposition of the derangement graph of S_n correspond to a normalization (integral form) of the hook-shaped *forgotten symmetric functions* $f_{(n-k,1^k)}$, which are dual to the elementary symmetric functions. In the language of assocation schemes, the integral form of f_λ has been dubbed *the combinatorial basis* in design-theoretic work of Martin and Sagan (see [10, Theorem 10]).

In his seminal monograph [9], Macdonald coined the term 'forgotten' symmetric functions since they had virtually no applications due to their unwieldy form [9, pg. 22]. Our results suggest that forgotten bases can provide simpler expressions for character sums over normal sets of elements with simple combinatorial/topological descriptions (e.g., t-derangements). Following Martin and Sagan's lead, we hope our work on the combinatorial basis kindles an iota of the interest that Macdonald's work has garnered, whose contributions to the theory of symmetric functions will surely never be forgotten.

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